

ON CONGRUENCE LATTICES OF PLANAR SEMIMODULAR LATTICES

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*To László Fuchs,
our teacher,
on his 90th birthday*

ABSTRACT. We prove that every finite distributive lattice D can be represented as the congruence lattice of a rectangular lattice K in which all congruences are principal. We verify this result in a stronger form as an extension theorem.

1. INTRODUCTION

In G. Grätzer and E. T. Schmidt [12], we proved that every finite distributive lattice D can be represented as the congruence lattice of a sectionally complemented finite lattice K . In such a lattice, of course, all congruences are principal, using the notation of G. Grätzer [8], $\text{Con } K = \text{Princ } K$.

Since every finite distributive lattice D can be represented as the congruence lattice of a planar semimodular lattice K (see G. Grätzer, H. Lakser, and E. T. Schmidt [11]), it is reasonable to ask whether instead of the sectional complemented lattice of the previous paragraph, we can construct a planar semimodular lattice K .

G. Grätzer and E. Knapp [9] proved a result stronger than the Grätzer–Lakser–Schmidt result: every finite distributive lattice D can be represented as the congruence lattice of a rectangular lattice K —see Section 2.1 for the definition. (For a new proof of this result, see G. Grätzer and E. T. Schmidt [15].) So keeping this in mind, we prove:

Theorem 1. *Every finite distributive lattice D can be represented as the congruence lattice of a rectangular lattice K with the property that all congruences are principal.*

We prove this representation result in a much stronger form, as an extension theorem.

Theorem 2. *Let L be a planar semimodular lattice. Then L has an extension K satisfying the following conditions:*

- (i) K is a rectangular lattice.
- (ii) K is a congruence-preserving extension of L .
- (iii) K is a cover-preserving extension of L .
- (iv) Every congruence relation of K is principal.

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Observe that we only have to prove Theorem 2. Indeed, let Theorem 2 hold and let D be a finite distributive lattice. By G. Grätzer and E. Knapp [9], there is a planar semimodular lattice K_1 whose congruence lattice is isomorphic to D . By Theorem 2, the lattice K_1 has a congruence-preserving extension K in which every congruence relation is principal. This lattice K verifies that Theorem 1 holds.

Let L be a planar lattice. An interior element of an interval of length two is called an *eye* of L . We will *insert* and *remove* eyes in the obvious sense. A planar semimodular lattice L is *slim* if it has no eyes.

We will use the notations and concepts of lattice theory as in [5].

For a deeper coverage of finite congruence lattices, see [4]. See G. Czédli and G. Grätzer [1] and G. Grätzer [6] for an overview of semimodular lattices, structure and congruences.

2. BACKGROUND

We need some concepts and results from the literature to prove Theorem 2.

2.1. Rectangular lattices. Let L be a planar lattice. A *left corner* (resp., *right corner*) of the lattice L is a doubly-irreducible element in $L - \{0, 1\}$ on the left (resp., right) boundary of L . A *corner* of L is an element in L that is either a left or a right corner of L . G. Grätzer and E. Knapp [9] define a *rectangular lattice* L as a planar semimodular lattice which has exactly one left corner, $\text{lc}(L)$, and exactly one right corner, $\text{rc}(L)$, and they are complementary—that is, $\text{lc}(L) \vee \text{rc}(L) = 1$ and $\text{lc}(L) \wedge \text{rc}(L) = 0$. In a rectangular lattice L , there are four boundary chains: the lower left, the lower right, the upper left, and the upper right, denoted by $C_{\text{ll}}(L)$, $C_{\text{lr}}(L)$, $C_{\text{ul}}(L)$, and $C_{\text{ur}}(L)$, respectively.

Let A and B be rectangular lattices. We define the *rectangular gluing* of A and B as the gluing of A and B over the ideal I and filter J , where I is the lower left boundary chain of A and J is the upper right boundary chain of B (or symmetrically).

We recap some basic facts about rectangular lattices (G. Grätzer and E. Knapp [9] and [10], G. Czédli and E. T. Schmidt [2] and [3]).

Theorem 3. *Let L be a rectangular lattice.*

- (i) *The ideal $\text{id}(\text{lc}(L))$ is the chain $C_{\text{ll}}(L)$, and symmetrically.*
- (ii) *The filter $\text{fil}(\text{lc}(L))$ is the chain $C_{\text{ul}}(L)$, and symmetrically.*
- (iii) *For every $a \leq \text{lc}(L)$, the interval $[a, \text{rc}(L) \vee a]$ is a chain, and symmetrically.*
- (iv) *For every $a \leq \text{lc}(L)$, L is a rectangular gluing of the filter $\text{fil}(a)$ and the ideal $\text{id}(\text{rc}(L) \vee a)$.*
- (v) *For every prime interval \mathfrak{p} of the chain $[a, \text{rc}(L) \vee a]$, there is a prime interval \mathfrak{q} of the chain C_{lr} so that \mathfrak{p} and \mathfrak{q} are perspective.*

Note that from (v) it follows that

$$\text{con}(C_{\text{ul}}) \leq \text{con}(a, \text{rc}(L) \vee a) \leq \text{con}(C_{\text{lr}}).$$

2.2. Eyes. Let L be a planar semimodular lattice. If we omit the interior elements of L in all intervals of length two, then we obtain a $\{0, 1\}$ -sublattice, $K = \text{Slim } L$. The elements of $L - K$ are called the *eyes* of L . A *slim* lattice has no eyes.

The slimming construction has a natural inverse. Let L be a planar semimodular lattice. Let L be slim. Let S be a covering square of L . Replace S by a copy of

the diamond M_3 . That is, we insert a new element, an *eye*, into S . We can obtain from lattice L lattice extensions K by *inserting eyes*.

2.3. Forks. We need from G. Czédli and E. T. Schmidt [3] the fork construction.

Let L be a planar semimodular lattice. Let L be slim. *Inserting a fork* to L at the covering square S , firstly, replaces S by a copy of N_7 . We get three covering squares replacing S .

Secondly, if there is a chain $u \prec v \prec w$ such that the element v has just been inserted (the element a or b in N_7 in the first step) and $T = \{x = u \wedge z, z, u, w = z \vee u\}$ is a covering square in the lattice L (and so $u \prec v \prec w$ is not on the boundary of L) but $x \prec z$ at the present stage of the construction, then we insert a new element y such that $x \prec y \prec z$ and $y \prec v$, see Figure 2. We get two covering squares to replace the covering square T .

Let K denote the lattice we obtain when the procedure terminates (that is, when the new element is on the boundary); see Figure 3 for an example.

The new elements form an order, called a *fork* (the black filled elements in Figure 3). We say that K is obtained from L by *inserting a fork* to L at the covering square S .

Here are some basic facts, based on G. Czédli and E. T. Schmidt [3], about this construction.

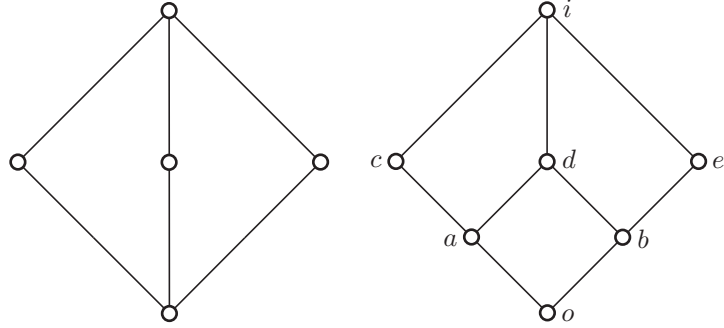


FIGURE 1. The lattices M_3 and N_7

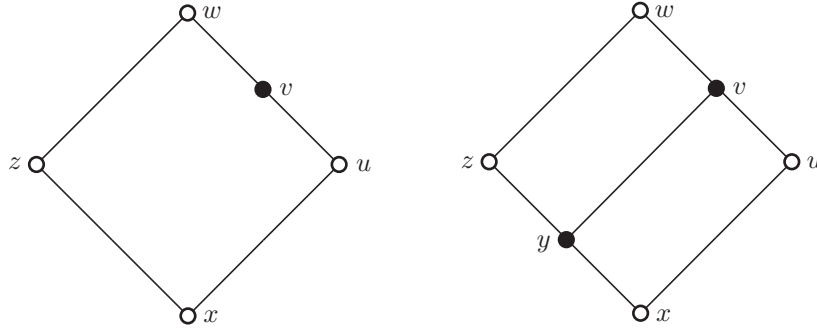
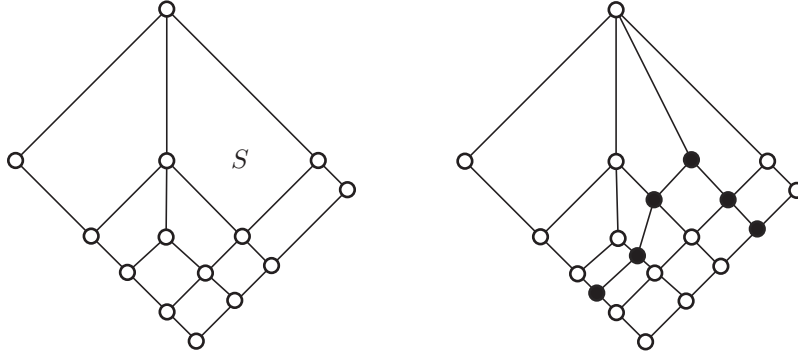


FIGURE 2. A step in inserting a fork

FIGURE 3. Inserting a fork at S

Lemma 4. *Let L be a planar semimodular lattice and let S be a covering square in L . If L is slim, then inserting a fork to L at S we obtain a planar semimodular lattice K . If L is rectangular, so is K .*

If y is an element of the fork outside of S , then $[y_, y]$ is up-perspective to $[o, a]$ or $[o, a]$, where y_* is the lower cover of y in K .*

2.4. Patch lattices. Let us call a rectangular lattice L a *patch lattice* if $\text{lc}(A)$ and $\text{rc}(A)$ are dual atoms; Figure 1 has two examples. The next lemma is a trivial application of Lemma 4.

Lemma 5. *Let L be a slim patch lattice and let S be a covering square in L . Inserting a fork to L at S , we obtain a slim patch lattice K .*

2.5. The structure theorem. Now we state the structure theorem for rectangular lattices of G. Czédli and E. T. Schmidt [3] (see also G. Grätzer [7]).

Theorem 6. *Let L be a rectangular lattice. Then there is a sequence of lattices*

$$K_1 = C_2^2, K_2, \dots, K_n = L$$

such that each K_i , for $i = 1, 2, \dots, n$, is either a patch lattice or it is the rectangular gluing of the lattices K_j and K_k for $j, k < n$.

2.6. A congruence-preserving extension. Finally, we need the following result of G. Grätzer and E. Knapp [9].

Theorem 7. *Let L be a planar semimodular lattice. Then there exists a rectangular, cover-preserving, and congruence-preserving extension K of L .*

3. THE PROOF

3.1. Congruences. To prove Theorem 2, we need a “coordinatization” of the congruences of rectangular lattices.

Theorem 8. *Let L be a rectangular lattice and let α be a congruence of L . Let α^l denote the restriction of α to C_{ll} . Let α^r denote the restriction of α to C_{lr} .*

Then the congruence α is determined by the pair (α^l, α^r) . In fact,

$$\alpha = \text{con}(\alpha^l \cup \alpha^r).$$

Proof. Since $\alpha \geq \text{con}(\alpha^l \cup \alpha^r)$, it is sufficient to prove that

(P) if the prime interval \mathbf{p} of L is collapsed by the congruence α , then it is collapsed by the congruence $\text{con}(\alpha^l \cup \alpha^r)$.

First, let L be a patch lattice. By Theorem ??, we obtain L from the square, C_2^2 , with a sequence of n fork insertions. We induct on n .

If $n = 0$, then $L = C_2^2$, and the statement is trivial.

Let the statement hold for $n - 1$ and let K be the patch lattice we obtain by $n - 1$ fork insertions into C_2^2 , so that we obtain L from K by one fork insertion at the covering square S .

We have three cases to consider.

Case 1. \mathbf{p} is a prime interval of K . Then the statement holds for \mathbf{p} and $\alpha|_K$, the restriction of α to K by induction. So \mathbf{p} is collapsed by $\text{con}((\alpha|_K)^l \cup (\alpha|_K)^r)$ in K . Therefore, (P) holds in L .

In the next two cases, we assume that \mathbf{p} is not in K .

Case 2. \mathbf{p} is perspective to a prime interval of K . Same proof as in Case 1. This case includes $\mathbf{p} = [o, a]$, any of the new intervals up-perspective with $[o, a]$ and symmetrically.

Case 3. $\mathbf{p} = [a, c]$, any of the new interval up-perspective with $[a, c]$ and symmetrically. Then the procedure defines the terminating prime interval $\mathbf{q} = [y, z]$ on the boundary of L which is up-perspective with \mathbf{p} , verifying (P).

Secondly, if L is not a patch lattice, we induct on $|L|$. By Theorem 6, L is the rectangular gluing of the rectangular lattices A and B over the ideal I and filter J . Let \mathbf{p} be a prime interval of L . Then \mathbf{p} is a prime interval of A or B , say, of A . (The case, \mathbf{p} is a prime interval of B is easier.) By induction, \mathbf{p} is collapsed by $\text{con}(\alpha|_{C_{\text{ll}}(A)} \cup \alpha|_{C_{\text{lr}}(A)})$, so it is collapsed by $\text{con}(\alpha|_{C_{\text{ll}}(L)} \cup \alpha|_{C_{\text{lr}}(L)})$. By Theorem 3.(v), every prime interval \mathbf{q} of $C_{\text{lr}}(A)$ is down-perspective to a prime interval of $C_{\text{lr}}(L)$, again (P) follows.

Finally, observe that (P) is preserved when inserting an eye. \square

3.2. Construction. Now we proceed with the construction for the planar semi-modular lattice L .

3.2.1. Step 1. We apply Theorem 7 to get a rectangular, cover-preserving, and congruence-preserving extension K_1 of K .

3.2.2. Step 2. Let $D = C_{\text{lr}}(K_1)$. We form the lattice D^2 , and insert eyes in the covering squares of the main diagonal, obtaining the lattice \widehat{D} , see Figure 4.

Now we do a rectangular gluing of K_1 and \widehat{D} , obtaining the lattice K_2 . Here is the crucial statement:

Lemma 9. *K_2 is a rectangular, cover-preserving, and congruence-preserving extension of K .*

For every join-irreducible congruence α of L , there is a prime interval \mathbf{p}_α of $C = C_{\text{ll}}(K_2)$ such that $\text{con}(\mathbf{p}_\alpha)$ in K_2 is the unique extension of α to K_2 .

Proof. Indeed, by Theorem 8, there is a prime interval \mathbf{q}_α^l of $C_{\text{ll}}(K_1)$ or a prime interval \mathbf{q}_α^r of $C_{\text{lr}}(K_1)$ such that $\text{con}(\mathbf{q}_\alpha^l)$ or $\text{con}(\mathbf{q}_\alpha^r)$ in K_1 is the unique extension of α to K_1 . If we have $\mathbf{q}_\alpha^l \subseteq C_{\text{ll}}(K_1) \subseteq C$, set $\mathbf{q}_\alpha^l = \mathbf{p}_\alpha$ and we are done.

If we have $\mathbf{q}_\alpha^r \subseteq C_{\text{lr}}(K_1)$ with $\text{con}(\mathbf{q}_\alpha^r)$ the unique extension of α to K_1 , then in K_2 there is a unique $\mathbf{q} \subseteq C_{\text{ll}}(\widehat{D}) \subseteq C_{\text{ll}}(K_2)$ such that in \widehat{D} , the prime intervals \mathbf{q}_α^r and \mathbf{q} are connected by an \mathbf{M}_3 on the main diagonal; see Figure 5 for an illustration.

Now clearly, we can set $\mathbf{p}_\alpha = \mathbf{q}$. \square

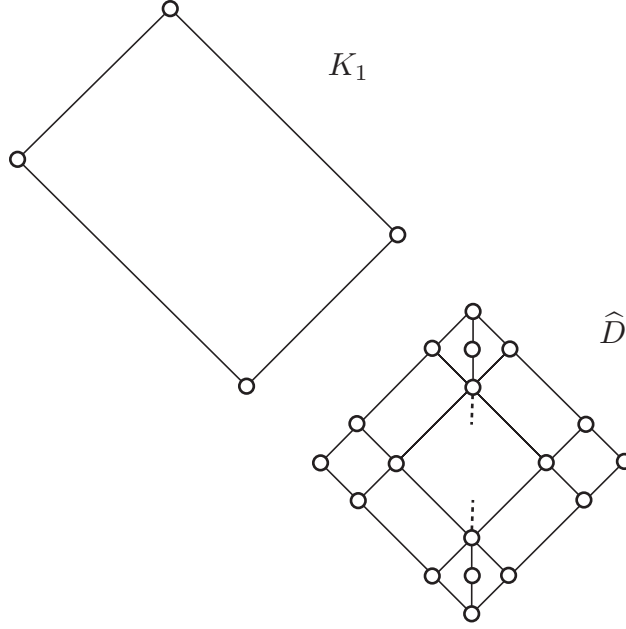
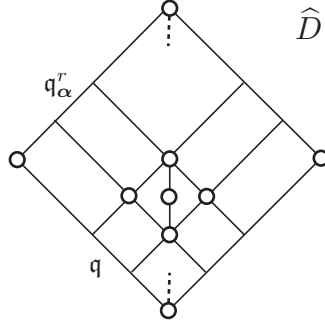


FIGURE 4. Step 2 of construction

FIGURE 5. Step 2 of construction: a detail of the lattice \hat{D}

3.2.3. *Step 3.* For the final step of the construction, take the chain $C = C_{\text{II}}(K_2)$ and a congruence α of L . We can view α as a congruence of K_2 and let $\alpha = \gamma_1 \vee \dots \vee \gamma_n$ be a join-decomposition of α into join-irreducible congruences. By Theorem 8 and (P), we can associate with each γ_i , for $i = 1, \dots, n$, a prime interval \mathfrak{p}_i of C so that $\text{con}(\mathfrak{p}_i) = \gamma_i$.

We construct a rectangular lattice $C[\alpha]$ (a cousin of \hat{D}) as follows:

Let $C_{n+1} = \{0 < 1 < \dots < n\}$. Take the direct product $C \times C_{n+1}$. We think of this direct product as consisting of n columns, column 1 (the bottom one), \dots , column n (the top one).

In column i , for $1 \leq i \leq n$, we take the covering square whose upper right edge is perspective to \mathfrak{p}_i and insert an eye. In the covering M_3 sublattice we obtain, every

prime interval \mathbf{p} satisfies $\text{con}(\mathbf{p}) = \gamma_i$. See Figure 6 for an illustration with $n = 3$; a prime interval \mathbf{p} is labelled with γ_i if $\text{con}(\mathbf{p}) = \gamma_i$.

Let b denote the top element of the M_3 we constructed for \mathbf{p}_n , clearly, $b \in C_{\text{ur}}(C[\alpha])$. Take the element $a \in C_{\text{ll}}(C[\alpha])$ so that the interval $[a, b]$ is of length n . Then the n prime intervals $\mathbf{q}_1, \dots, \mathbf{q}_n$ of $[a, b]$ satisfy

$$\text{con}(\mathbf{q}_1) = \gamma_1, \dots, \text{con}(\mathbf{q}_n) = \gamma_n,$$

so $\text{con}([a, b]) = \alpha$, finding that in the lattice $C[\alpha]$, the congruence α is principal.

We identify C with $C_{\text{ur}}(C[\alpha])$; note that this is a “congruence preserving” isomorphism: for a prime interval \mathbf{p} of C , the image \mathbf{p}' of \mathbf{p} in $C_{\text{ur}}(C[\alpha])$ satisfies $\text{con}(\mathbf{p}) = \text{con}(\mathbf{p}')$.

Now we form the rectangular gluing of $C[\alpha]$ with filter C and K_2 with the ideal C to obtain the lattice $K_2[\alpha]$. Obviously, $K_2[\alpha]$ is a rectangular lattice, it is a cover-preserving congruence-preserving extension of K_2 and, therefore, of L .

Observe that $C_{\text{ll}}(K_2[\alpha])$ is still (congruence) isomorphic to C , so we can continue this expansion with all the congruences of L . In the last step, we get the lattice $K_3 = K$ satisfying all the conditions of Theorem 2.

3.3. Discussion. Let L be a rectangular lattice and let α be a join-irreducible congruence of L . We call α *left-sided*, if there a prime interval $\mathbf{p} \subseteq C_{\text{ll}}(L)$ such that $\text{con}(\mathbf{p}) = \alpha$ but there is no such $\mathbf{p} \subseteq C_{\text{lr}}(L)$. In the symmetric case, we call α *right-sided*. The congruence α is *one-sided* if it is left-sided or right-sided. The congruence α is *two-sided* if it is not one-sided.

Using these concepts, we can further analyze Theorem 8 and condition (P). By G. Czédli and E. T. Schmidt [3] (see also G. Grätzer [7]), we build a rectangular lattice from a grid (the direct product of two chains) by inserting forks and eyes. At the beginning, all join-irreducible congruences are one-sided. When we insert a fork, we introduce a two-sided congruence. When we insert an eye, we identify two congruences, resulting in a two-sided congruence.

What congruence pairs occur in Theorem 8? Let β_l be a congruence of $C_{\text{ll}}(L)$ and let β_r be a congruence of $C_{\text{lr}}(L)$. Under what conditions is there a congruence α of L such that $\alpha^l = \beta_l$ and $\alpha^r = \beta_r$? Here is the condition: If \mathbf{p} is a prime

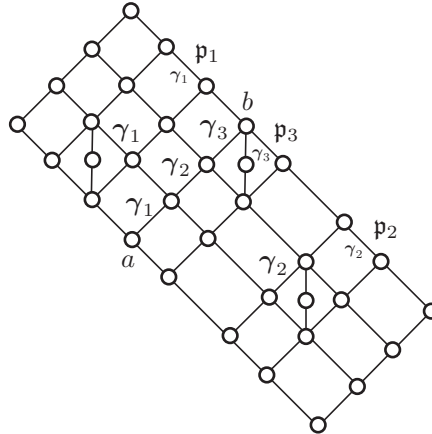


FIGURE 6. Step 3 of construction: the lattice $C[\alpha]$

interval of $C_{\text{II}}(L)$ collapsed by β_i and there is a prime interval \mathfrak{q} of $C_{\text{Ir}}(L)$ with $\text{con}(\mathfrak{p}) = \text{con}(\mathfrak{q})$, then \mathfrak{q} is collapsed by β_r ; and symmetrically.

In Step 3 of the construction, we use the chain C_{n+1} . Clearly, C_n would have sufficed. Can we use, in general, shorter chains?

In a finite sectionally complemented lattice, the congruences are determined around the zero element. So it is clear that for finite sectionally complemented lattices, all congruences are principal.

For a finite semimodular lattice, the congruences are scattered all over. So it is somewhat surprising that Theorem 1 holds.

For modular lattices, the situation is similar to the semimodular case. E. T. Schmidt [17] proved that every finite distributive lattice D can be represented as the congruence lattice of a countable modular lattice K . (See also G. Grätzer and E. T. Schmidt [13] and [14].) It is an interesting question whether Theorem 1 holds for countable modular lattices.

Patch lattices are the basic building stones of rectangular lattices; indeed, it was proved by G. Czédli and E. T. Schmidt [3] that every patch lattice can be obtained from the four-element Boolean lattice by a sequence of insertions of first forks and then eyes, and conversely. It would be interesting to use this result to characterize congruence lattices of patch lattices.

REFERENCES

- [1] G. Czédli and G. Grätzer, *Planar Semimodular Lattices: Structure and Diagrams*. Chapter in [16].
- [2] G. Czédli and E. T. Schmidt, *Slim semimodular lattices. I. A visual approach*, Order **29** (2012), 481-497.
- [3] ———, *Slim semimodular lattices. II. A description by patchwork systems*, Order.
- [4] G. Grätzer, *The Congruences of a Finite Lattice, A Proof-by-Picture Approach*. Birkhäuser Boston, 2006. xxiii+281 pp. ISBN: 0-8176-3224-7.
- [5] G. Grätzer, *Lattice Theory: Foundation*. Birkhäuser Verlag, Basel, 2011. xxix+613 pp. ISBN: 978-3-0348-0017-4.
- [6] G. Grätzer, *Planar Semimodular Lattices: Congruences*. Chapter in [16].
- [7] ———, *Notes on planar semimodular lattices. VI. On the structure theorem of planar semimodular lattices*. Algebra Universalis.
- [8] ———, *The order of principal congruences of a lattice*. arXiv 1302.4163.
- [9] G. Grätzer and E. Knapp, *Notes on planar semimodular lattices. III. Rectangular lattices*. Acta Sci. Math. (Szeged) **75** (2009), 29–48.
- [10] ———, *Notes on planar semimodular lattices. IV. The size of a minimal congruence lattice representation with rectangular lattices*. Acta Sci. Math. (Szeged) **76** (2010), 3–26.
- [11] G. Grätzer, H. Lakser, and E. T. Schmidt, *Congruence lattices of finite semimodular lattices*. Canad. Math. Bull. **41** (1998), 290–297.
- [12] G. Grätzer and E. T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.
- [13] ———, *On finite automorphism groups of simple arguesian lattices*, Studia Sci. Math. Hungar. **35** (1999), 247–258.
- [14] ———, *On the Independence Theorem of related structures for modular (arguesian) lattices*. Studia Sci. Math. Hungar. **40** (2003), 1–12.
- [15] ———, *A short proof of the congruence representation theorem of rectangular lattices*. arXiv: 1303.4464. Algebra Universalis (2013).
- [16] G. Grätzer and F. Wehrung eds., *Lattice Theory: Empire. Special Topics and Applications*. Birkhäuser Verlag, Basel.
- [17] E. T. Schmidt, *Every finite distributive lattice is the congruence lattice of some modular lattice*. Algebra Universalis **4** (1974), 49–57.

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